Complete Reducibility of Representations

Prop: Every non-0 representation of a finite group G is either irreducible or decomposable.

Theorem (Maschke): $|G| < \infty$

Every finite dimensional representation of G is completely reducible

Proof: Induction on dimension of φ

$\dim \varphi = 1 \implies$ irreducible.

$\dim \varphi > 1 \implies$ irreducible, or

$$\varphi \sim \underbrace{\varphi^{(1)}}\oplus\underbrace{\varphi^{(2)}}, \quad \dim \varphi^{(k)} < \dim \varphi$$

$k = 1, 2$

completely reducible by induction

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Rem: Uses $|G| < \infty$, $\dim \varphi < \infty$,
$F (= \mathbb{C})$ has characteristic 0

e.g. $\varphi : \mathbb{Z} \to GL_2(\mathbb{C})$, $\varphi_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

or

$\varphi' : \mathbb{Z}_p \to GL_2(\mathbb{Z}_p)$, $\varphi'_{[k]} = \begin{bmatrix} [1] & [k] \\ [0] & [1] \end{bmatrix}$

p prime

are not completely reducible.

Cor: $\varphi : G \to GL_n(\mathbb{C})$, then
$\exists$ $\underline{P} \in GL_n(\mathbb{C})$ such that

$$P^{-1} \varphi P = \begin{bmatrix} \varphi^{(1)} & 0 & & 0 \\ 0 & \varphi^{(2)} & & 0 \\ & & \ddots & \\ 0 & 0 & & \varphi^{(r)} \end{bmatrix}$$

where $\varphi^{(k)} : G \to GL_{n_k}(\mathbb{C})$
which is irreducible.

Example: $\rho: S_3 \longrightarrow GL_3(\mathbb{C})$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = [\,\underbrace{v}_{W} \quad \underbrace{x \quad y}_{U}\,]$$

$$P^{-1}\varphi P = \left[\begin{array}{c|cc} \varphi^{(1)} & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\varphi^{(2)}$}} \\ 0 & & \end{array}\right]$$

$$P^{-1}\varphi_{(12)}P = \left[\begin{array}{c|cc} 1 & & \\ \hline & -1 & 1 \\ & 0 & 1 \end{array}\right]$$

$$P^{-1}\varphi_{(123)}P = \left[\begin{array}{c|cc} 1 & & \\ \hline & 0 & -1 \\ & 1 & -1 \end{array}\right]$$